Philosophical questions and didactical considerations on a reality-oriented mathematics education

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Abstract

We observe a growing trend towards examples and tasks that promise more relevance to reality, more learning for life in teaching mathematics. We therefore see an increasing need to think through the terms used in this context in the relevant publications, such as reality or modelling, also theoretically in the context of mathematics didactics. This need arises both from requirements for the quality of the work performed and with regard to the scientific nature of mathematics didactics in the eyes of other sciences[1].

Since the time of the ancient Greeks, philosophy has offered a multitude of – contradicting – epistemological considerations as a theoretical foundation. In modern times, controversial theoretical and sociological theories have emerged. For this contribution to the discussion, we have made a small selection of theories that are relevant from our point of view in order to set a starting point. In the last Section on the epistemological view of models and modelling, we develop a new theory about mathematical modelling in school.

Keywords:

Reality

Philosophy

Modelling

Teaching mathematics

Mathematics education

[1] According to Niklas Luhmann, a science can be assessed by the quality of its activity in the three system references function (research), performance (education) and reflection (communication about its own foundations) (Luhmann, 2017, pp. 471 – 484). With this contribution, we hope to give an impulse to our science, mathematics didactics, in the field of reflection.

1 Mathematics, reality and truth: the question of recognition

In the eyes of many people, mathematics is a guarantee of truth. Not only natural sciences, but also social sciences and humanities rely on mathematical research methods. In political and social discourses, we also often see mathematical arguments, for example in models or statistics on climate development or on an eco-tax or a motorway toll presented in convincing graphics. The desire to be able to rely on something very firmly and securely is psychologically very understandable. However, the "education towards maturity" or the ability to criticize, which is rightly required in all curricula, requires us to make it clear to ourselves and – more important! - to our students in what limited inner-mathematical sense the statements of the science of mathematics are "true": "Pure mathematics is just statements that say so and so is true if such and such is true of something. It is essential that it is not even discussed whether the first statement is actually true, nor is it mentioned what the something is to which it presumably applies…" (Russell, 1901, p. 8 – see more about this in Section 2.3.) What is the relation of this game of statements to our real world?

In the Austrian law for general aims of learning mathematics we found this aim: "Epistemological aspect: Mathematics is a special way of recognizing our world of experience. It is a specific way of perceiving the phenomena of the world and understanding them through abstraction. Mathematizing a real phenomenon can significantly deepen everyday experience." (NOSt, 2018, Aspects of mathematics).

In Germany, the curricula in connection with the acquisition of knowledge usually immediately mention modelling, in Bavaria it says: "The central task of mathematics lessons at high schools is therefore to give the students not only specific mathematical knowledge and working methods but also more general insights into processes of thought and to convey the decision-making process that is important for actively and responsibly shaping society. It becomes clear to the young people that mathematics can be a helpful tool for analysis and for gaining knowledge…" (ISB Bavaria, n. d.).

We are convinced that in your country the mathematics curriculum includes the aim that students should learn something about mathematics and recognition and the relations of mathematics, reality and truth.

From our point of view, it is very important in this context that the "truth" of applications of mathematics, such as economic models or technical model calculations, is by no means automatically secured by proven mathematical statements. We distinguish the "truth" of formal-axiomatic pure mathematics (cf. Section 2) from the "truth" (actually better: usability, predictability, empirically proven realism etc. - cf. Section 5) of various applications of mathematics. To make a didactic appeal right here: Today, model criticism is an important part of the required critical ability of responsible citizens. We are therefore pleased to note that many didactic aspects of this critical ability, for example in the area of statistics and data (see, for example, the online archive of the journal "Stochastics in School"[2]), have already been

[2] https://www.stochastik-in-der-schule.de/sisonline/index.htm. Accessed: 21.05.2022

discussed and implemented in teaching proposals (cf. the mathematics lesson proposals collected by “MUED” teachers - see www.mued.de).

According to our knowledge of history and anthropology, at all times people have been concerned with questions about reality, its coming into being and passing away, its meaning and our possibilities to influence it. In the search for convincing answers, there have often been and still are smooth transitions to religions, from belief in a nature with a soul to belief in an almighty being who created everything. Why is certainty so important? If we know the truth about something, we can be sure and save. If we are sure that not car is coming, we can cross the road without risk. If we know the truth about vaccination, we can go to a doctor and be protected without fear. If we know that we are the reason for dramatic climatic change and its negative consequences we can start (or go on) change us and our way of life. If we are not able to decide what fake news is and what truth is, we will have to believe in people that say they know it all. Which of all the people that try to influence us is the one we can trust? Many people have bad luck when they select one or some of them. In this situation, our massage is most important: Mathematics is very helpful to find out the fake news and to identify the false prophets! However, there must be some change in teaching in learning mathematics to use mathematics in this way.

Some more knowledge about history and philosophy of mathematics is very useful in this context. Did you know this? "The awe-inspiring amazement at the logical consistency and power of mathematical knowledge, what so close to mystical experience – so close that not a few of the deepest thinkers crossed the line: PYTHAGORAS, KEPLER, NEWTON, to a certain extent EINSTEIN. – The anticipation of a hidden ideal reality to which mathematical thinking grants us access: PLATON; – but which is only fully recognizable to a superhuman, divine intelligence: GAUSS (a thought that celebrated a surprising resurrection in the “Platonism” of modern mathematical philosophy, in the midst of the sober scepticisms of the 20th century, and free from religious garb). – Mathematics as a guarantee of human reasonableness: VOLTAIRE; – and the exaggeration of the spiritual longing contained therein to a desperate clinging to mathematical certainty as a pledge of absolute truth: HILBERT, […]” (Wittenberg, 1963, p. 49).

Two philosophers whose opposing views have played a more or less important role in the history of mathematics are particularly relevant to the ontological question in mathematics: Plato and Aristotle (cf. for example Reichel, 1988).

2 Are we discovering or inventing mathematics?

2.1 Plato and Aristotle

According to Plato (~ 400 BC), mathematicians only discover what already exists in the world of ideas. He wrote that the geometers, astronomers and arithmeticians do not invent their figures and other symbols, but they only investigate what is already there (Plato, 1994, Euthydemos 290b).

Socrates explains “that they (geometers, JM and SG) use the visible figures and always talk about them, while the real object of their thinking is not these, but those of which these are mere images. It is the square for itself and the diagonal itself, about which they make their deliberations, but not that which they sketch out" (Plato, 1994, The State 511a). Here is an opportunity to address the difference between ideas and their visible forms in the classroom. Ask your students to draw two points *A* and *B* and connect them with the shortest line named *s*. This is how the line segment is created. What do the students do? You mark two dots on the paper, either as blobs or as small crosses, and connect them with a ruler. Can you then see the points of the line segment? Of course not! A point, according to Euclid's definition, is something without extension, indivisibly small. A line has only one dimension, length. If you see a line on the paper or on the blackboard, it is only possible because you are seeing an area, which is probably a rectangle, which has two very short sides (the width of the line) and two long sides – the connection between the two points. Therefore, speaking correctly, you are only allowed to set the task: Draw an image of the idea of two points *A* and *B* and their connection with the shortest line "*s*". This is new for many geometry lessons and a good starting point to learn something about the philosophical background of mathematics.

Plato's view has further major effects on the claimed or accepted truth content and reality reference of mathematical theories, as W. Stegmüller explains: "According to the classical (Platonist) view, every meaningful mathematical statement is true or false, even if for us (so far or maybe forever) there is no way of deciding whether one or the other is the case; the mathematical facts exist independently of their being recognized or not being recognized by us. According to this, the progress of mathematical knowledge basically consists in the fact that it is established for an increasingly large class of statements that they are true sentences, these are theorems, which are in coincidence with the mathematical facts that exist 'in themselves'. The research activity of the mathematician is similar to that of a discoverer: analogous to how a biologist discovers previously unknown plants and animals when exploring a new stretch of land, the mathematician discovers new concepts and relationships between them, which he records in axioms and theorems." (Stegmüller, 1978, p. 675).

Formal-axiomatic mathematics, as practiced in academic research at universities, is very close to Platonism. We will come back to this in section 2.3. For modelling, the application of mathematics in real-world mathematics teaching, we cannot accept that all ready-made mathematical models already exist in the world of ideas. In mathematics lessons we can sometimes live with the fact that descriptive models want to describe and explain nature as it is (as it is in the world of ideas, or as God created it, or as it developed in the course of evolution). In the case of normative models, for example on taxes or other regulations and their possible effects, we need a different basic philosophical theory, for example from the scientific-sociological or Marxist spectrum (Sections 2.4 and 2.5).

The traditional counter-position to Plato we call today "constructivist[3]" is going back to Aristotle (~ 350 BC): "Aristotle, on the other hand, understands mathematics as a product of abstraction from empirical reality, to which it can therefore also be applied again", the Lexicon of Mathematics under the heading "Philosophy of Mathematics" says with reference to Aristotle[4].

[3] Such allocations are, of course, simplifications!

[4] https://www.spektrum.de/lexikon/mathematik/philosophie-der-mathematik/7823. Accessed: 21.05.2022

We find traces of this view in intuitionist mathematics [e.g. in L. E. J. Brouwer (1881 – 1966), cf. Section 2.3] and constructive philosophy of science [e.g. in E. v. Glasersfeld (1917 – 2010) or in P. Lorenzen (1915 – 1994)], but also in Ludwig Wittgenstein (1889 – 1951): "The mathematician is an inventor, not a discoverer." (Wittgenstein, 1974, § 168, p. 99).

2.2 Kant and the philosophical basics of geometry

We continue our journey through the history of theory with I. Kant (1724 – 1804): “Judgments may have an origin which they want, or be constituted according to their logical form as they want, there is still a difference according to their contents, by virtue of which they are either merely explanatory and do not add anything to the content of the knowledge, or expand and enlarge the given knowledge; the first are called analytical, the second synthetic judgments.” (Kant, 1781/1979, p. 14 f.). Kant characterized mathematics as a collection of "synthetic judgments a priori". New knowledge is therefore a work of man, a synthesis.

Kant’s prominence also had a downside, a kind of ban on thinking, which affected C. F. Gauss (1777 – 1855) very much: “I also occasionally have individual free hours used for thinking about another topic that I have been dealing with for almost 40 years thought again, I mean the first foundations of geometry [...] In the meantime I will probably not get to work out my very extensive investigations into public notice for a long time, and perhaps it will never happen in my lifetime, since I will shy at the clamour of the Boeotians if I fully express my opinion.” (quoted from Wußing, 1982, p. 52 f.). A. Heyting (1898 – 1980) wrote on the relationship between mathematics and metaphysics: "We have no objection against a mathematician privately admitting any metaphysical theory he likes [...] In fact all mathematicians [...] are convinced that in some sense mathematics bear eternal truth, but when trying to define precisely this sense, one gets entangled in a maze of metaphysical difficulties. The only way to avoid them is to banish them from mathematics.” (quoted from Meschkowski, 1965, p. 33). Gauss was convinced: "It is precisely in the impossibility of deciding between [...] Euclidean and non-Euclidean geometry a priori that there is clear proof that Kant was wrong to claim that space is only a form of our intuition." (cited after Wußing, 1982, p. 56). However, Gauss did not publish his results.

N. I. Lobatschewski was the first who publicly present his research on non-Euclidean geometry[5] in a lecture at the University of Kazan on February 11, 1826. He justified his efforts in a remarkably modern way: "Be that as it may, the new geometry for which the basis is now laid, if it does not exist in nature, can nevertheless exist in our imagination, and even if it is not used in real measurements, it will open up a new, wide field for the application of geometry and analysis to each other.” (Lobatschewski, 1899, p. 83).

2.3 Cantor and the crisis of philosophical foundations of mathematics: formal and real in mathematics

With this argument, Lobatschewski points to a point of contention that is very important for the further development of mathematics. In the history of mathematics in modern times up to the 19th century, natural science and mathematics went hand in hand, mathematicians were not “pure” mathematicians in a modern sense. With non-Euclidean geometry, paths

[5] An introductory work on this topic is, for example, Baldus and Löbell (1964).

began to separate. Other milestones were certain phenomena with infinity in analysis and in particular G. Cantor's "set theory". The formal as the justification of existence found its way into mathematics. The dispute revolved around the question of whether it is permissible to define and examine purely formally derivable but not intuitively conceivable mathematical objects such as "pathological" functions (e.g. in Boese and Luther, 1981) or certain defined infinite (transfinite) sets defined by G. Cantor (1845 – 1918), (cf. Volkert, 1997). The statement that the natural numbers $\mathbb{N}$ are equal to the even numbers $\mathbb{N}_g$ can be justified very early on in mathematical education: the function f: $\mathbb{N} \mapsto \mathbb{N}_g$, $f(n) = 2n$ is bijective. Alternatively, Hilbert's hotel with a countable number of rooms that is fully occupied: if another potential guest appears, he or she can be accommodated without further ado. The hotel owner puts him or her in room no. 1, the guest from room no. 1 in room no. 2, etc. This means that every guest is accommodated in a room (cf. Heuser, 1986a, p. 139).

“We can speak of the reality or existence of integers in two senses, finite and infinite [...] Firstly, we may regard integers as real insofar as they have a very specific place in our mind on the basis of definitions be distinguished from all the other components of our thinking in the best possible way, stand in a certain relationship to them and thus modify the substance of our mind in a certain way [...] But then reality can also be ascribed to the numbers it can be considered that they represent an expression or an image of processes and relationships in the exterior world facing the intellect [...] With the thoroughly realistic, but at the same time no less idealistic basis of my considerations, I have no doubt that these two types of reality always come together.” (Cantor, 1932, p. 181).

According to Heyting, however, the existence of mathematical objects is “only secured insofar as they can be determined by thinking; they only have properties insofar as these can be recognized by thinking about them. However, this possibility of knowledge is only revealed to us through knowledge itself. Belief in transcendent existence, which is not supported by concepts, must be rejected if it supposed as mathematical evidence.” (Heyting, 1934/1974, p. 106 f.)

We quote L. Kronecker (1823 – 1891) as a small indication that mathematicians also engaged in emotional debates. He called G. Cantor a "spoiler of youth" (quoted from Meschkowski, 1973, p. 41). A. Mostowski (1913 – 1975) expressed himself in a less extreme way: “The only consistent point of view, which is in accordance with both common sense and the mathematical tradition, is […] the assumption that the origin and ultimate 'raison d'être' (reason for being) of the concept of number, both natural and real, lie in experience and in practical applicability.” (quoted from Meschkowski, 1965, p. 32).

In retrospect, the (Platonic-formalistic[6]) group around Bourbaki prevailed. Within their own science of mathematics – separate from the natural sciences – they themselves have set the rules for what is allowed and what is not permitted, what is and what is not recognized as mathematics. However, it must also be pointed out at this point that some mathematical theories, which were still considered purely formal and unrelated to application at that time,

[6] Here, too, we are shortening an extensive discussion by assigning or labelling.

have now opened up surprising and helpful fields of application in natural sciences and technology (cf. Reichel and Zöchling, 1990).

Another historically very important occasion for public debate about the nature, the possible justification, the truth and the meaning of mathematics was G. Cantor's "set theory" – more precisely: an error in the basic definition: "Under a 'set' we take any combination *M* of certain well-distinguished objects in our intuition or thought (which are called the 'elements' of *M*) into a whole" (Cantor, 1932, p. 282).

B. Russell (1872 – 1970) studied the set *M* of all sets that do not contain themselves as an element. This set includes *B*. the natural numbers, the real numbers and much more. a. Now is *M* itself an element of *M*? Suppose *M* is an element of *M*. It follows, by definition, that *M* has the property: *M* is not an element of *M*. However, assuming that *M* is not an element of *M*, it follows precisely from this property that *M* is an element of *M*. Since *M* can be neither an element nor an element of *M*, the error must lie in the definition of *M*. However, the definition of the set *M* is correct according to Cantor's definition of a set – that is why the error is already in Cantor's definition (cf. Russell, 1903).

Russell's attempt to found mathematics differently is called "Logicism". He formulated a pointed statement about mathematics: "Pure mathematics consists merely of assertions which state that such and such a proposition applies if such and such a statement applies to something. It is essential that it is not even discussed whether the first statement is actually true, and that it is also not mentioned what the something is to which it presumably applies [...] In pure mathematics we start from certain derivation rules, by which we can conclude that if one proposition is true, then a second proposition is also true. These derivation rules form the main part of the principles of formal logic. Therefore, we choose any hypothesis that we find amusing and derive the appropriate conclusions. If this hypothesis applies to something and not to one or more individual facts, then our deductions constitute mathematics. We can call mathematics as that area in which we never know what we are actually talking about and whether what we say is actually true." (Russell, 1901/1967, p. 8 f.).

Russell's attempt at justification failed – instructively – as did other formalist approaches. Here we would like to hint at D. Hilbert's (1862 – 1943) proposal, which was probably the most momentous for the further development of mathematics, but also failed from a mathematical point of view (to Gödel!): "Everything that constitutes mathematics in the previous sense is strictly formalized, so that proper mathematics, or mathematics in the narrower sense, becomes a body of formulas. In addition to the actual [...] formalized mathematics, there is a kind of new mathematics, a metamathematics that is necessary to secure – in contrast to the purely formal methods of reasoning in actual mathematics – the content-related reasoning, but only to prove the consistency of the axioms. In this metamathematics, it will be operated on the proofs of the actual mathematics, and these latter themselves form the object of the content-related investigation. In this way, the development of mathematical science as a whole takes place in two different ways: by obtaining new provable formulas from the axioms by means of formal reasoning and on the other hand by adding new axioms together with the proof of consistency by means of reasoning with regard to content." (Hilbert, 1923, p.180).

Important for the further development of mathematics was the proposed division of labour into the vast majority of mathematicians who “do” mathematics and a small group of specialists in basic research who think about mathematics and logic. For the vast majority, this division of labour means a significant reduction in workload, since they do not have to worry about the difficult and still unresolved basic problems. From this point of view, this "solution" is expressly welcomed: "It would be of no interest to us (Bourbaki, JM and SG) to pursue even roughly the incessant and impassioned polemics that this problem raises, which is unusually favourable to a breeding ground of metaphysical or theological fantasies. We only want to go into the point of view that the majority of mathematicians have held since antiquity. It essentially consists in refusing the dispute, since it cannot be settled irrefutably” (Bourbaki, 1971, p. 39, cf. Meschkowski, 1965, p. 33). From a philosophical point of view, this is obviously unsatisfactory. C. Thiel calls this attitude "provocatively hostile to logic and fundamentals" (Thiel, 1972, p. 128). From the perspective of the sociology of science, on the other hand, a division of labour is an obvious solution to the problem (cf. Maaß, 1988).

With regard to the truth and reality of mathematics, it is particularly important to note that the current use of axioms in mathematics goes back to Hilbert and Bourbaki. In contrast to those of Euclid, these axioms do not claim to fix reality in a plausible way – they are, as Russell emphasized, in a certain sense arbitrary, set by people and essentially justified by the fact that they are free of contradictions and they form a good starting point for the sentences deduced from them. The Zermelo-Fraenkel set theory with axiom of choice has turned out to be an extremely successful starting point for the derivation of mathematical theorems. According to Gödel, however, a proof of the consistency of the axiom system cannot be provided (second incompleteness theorem). With regard to the initial question about the truth of mathematics, we can state as an interim conclusion that formal-axiomatic mathematics is only "true" if one believes in the validity of the basic axioms (for which there are many good reasons, but not in the absolute sense that they are "true")[7].

A non-formal proposal for solving the fundamental crisis triggered by the antinomies is a solution to the problem from a philosophical point of view, but was strictly rejected by the mathematicians because, in their opinion, the proposal was too restrictive and excludes only formally justifiable areas of mathematics. According to this proposal, a set should be defined as follows: "A set is a law by virtue of which, if an arbitrary number is elected over and over again, each such choice produces either a particular series of characters, with or without terminating the process, or the inhibition of the process together with the definitive destruction of its result, whereby for each $n > 1$ after each unfinished and uninhibited sequence of $n - 1$ elections, at least one number can be specified which, if chosen as the n^{th} number, does not cause the inhibition of the process. Every sequence of characters generated in this way from an unlimited sequence of choices (which is therefore in general not fully representable) is called an element of the set. We will briefly call the common type of type of generation of the elements of a set *M* also as the set *M*.” (Brouwer, 1925, p. 244 f.).

[7] For mathematics education, the formal-axiomatic point of view has not proved fruitful. The New Math movement, which was launched after the Sputnik shock, failed after only a few years and has been replaced, among other things, by an application-oriented approach (cf. Tietze et al., 2000, Sections 1.1.1 and 1.1.2).

It is not difficult to notice in this definition that a quantity is constructed here element by element. This variant of Aristotle's line is called "intuitionism": "Mathematics is a free creation independent of experience; it develops from a single simple intuition a priori (a priori intuition)." (Brouwer, 1907, p. 179). Like Brouwer, Heyting also writes: "Mathematical objects are grasped directly by the thinking mind; mathematical cognition is therefore independent of experience." (Heyting, 1934/1974, p. 3).

2.4 A materialistic view

Intuitionism has not only been rejected by the vast majority of mathematicians for practical reasons, but has also been criticised from a completely different direction, which we would like to discuss briefly here because it rounds off the spectrum of answers to the question of the nature of mathematics and opens up a perspective for a new philosophical basis for modelling. "The mental constructions of which the intuitionists speak have no relation to the real world and therefore possess a purely subjective character." (Rusavin, 1968, p. 249). From the point of view of historical materialist philosophy, the purely subjective character is not acceptable, as A. A. Markov (1856 – 1922) wrote: "I can by no means agree to 'intuitive clarity' being considered as a criterion of truth in mathematics, because this criterion means the complete triumph of subjectivism and is incompatible with the conception of science as a form of social activity." (quoted in Rusavin, 1968, p. 262). Markov thus refers to a basic position of historical materialism: "Labour is the source of all wealth [...] But it is infinitely more than this. It is the first basic condition of all human life, and to such a degree that we must in a certain sense say: it created man himself." (Engels, 1978, p. 444).

Incidentally, Engels (1820 – 1895) also commented directly on mathematics: "The mystery that still surrounds the quantities used in the infinitesimal calculus – the differentials and infinities of various degrees – is the best proof that one still imagines that one is dealing here with pure 'free creations and imaginings' of the human spirit, for which the objective world offers no equivalent. Yet the opposite is the case. Nature offers the models for all these imaginary quantities." (Engels, 1978, p. 530).

P. Ruben (*1933) has formulated a precisely thought-out version of the historical-materialist view of mathematics in his works. Its starting thesis is: "For Marxist-Leninist philosophy, therefore, mathematics becomes a philosophical object when it is thought of as a process of production." (Ruben, 1979, p. 69). This brings him finally to the definition: "If general labour becomes production and calculation of abstract values, it becomes mathematics." (Ruben, 1979, p. 86). From this point of view, it is then not just a happy coincidence that even very abstract mathematics has and retains a relation to the real world. For modelling in reality-based mathematics teaching, this also opens up a philosophically well-founded way to avoid the typical dualism problems associated with how Blum and Kaiser make a distinction between mathematics and reality (or the rest of the world, cf. Blum and Kaiser (2018, p. 3 f.)).

The historical materialist view of the relationship between mathematics and reality, which we will only briefly mention here for reasons of space, as well as all the previously mentioned philosophical views, but which we will by no means discuss or evaluate, is not the only one that addresses a social aspect of the relationship between mathematics and reality[8]. Therefore, a section on the sociology of science follows now.

2.5 Sociology of science

It is already clear from the few quotations on the development of mathematics in the 19th century in this text that the people involved, such as Gauss, Kronecker, Cantor etc., were people who more or less consciously made their personal religious and philosophical convictions to the basis of their mathematical work and defended them. In addition, mathematicians set standards as (a) group(s), as a scientific association, as a team of editors or reviewers of a journal or a book, as a board of professors at a university who decide on academic qualifications and appointments etc. They make decisions about what is accepted as a scientific contribution to mathematics and what is not. According to the current state of research in the sociology and history of science, no one can claim without contradiction that mathematics has developed objectively and independently of the people involved. An example that can certainly be used in university teaching is the well-documented work of the mathematician Hermann Graßmann (1809 – 1877), whose "theory of extension", long ignored by experts, formed the basis for today's linear algebra (Tietze, 2000, p. 84 ff.). An important connection between mathematics and reality (especially the professional world in our society) was and is those people who pursue mathematics as a profession, be it at a university, at a research institution funded by and for business or for the military, be it with the job title "mathematician" or another from the STEM field or the social and economic sciences[9]. In other words, mathematics does not appear in social reality by itself, people who earn money with it use it for various purposes. These purposes are by no means always value-free, objective, just, socially fair, etc. The simple question of who can afford to pay mathematicians in order to better pursue their own goals leads to the suspicion of partiality: Those who pay determine the direction of research. The client determines which data should be taken into account and what is considered the optimum to be achieved – and usually not the people concerned and not the contractors. When an issue is mathematically modelled with different intentions, very different models often emerge. The controversy over a motorway toll in Germany shows this well – even with school mathematics. The topic of "income tax" also fits in this context (Henn, 2006).

With less simple words, the discussion about "knowledge and interest" was opened many years ago by Jürgen Habermas (Habermas, 1973), and the changed and now by no means value-free and objective role of science in social conflicts was analysed by Robert Tschiedel using the example of nuclear energy (Tschiedel, 1977). Today's status of the conflict over the role and "freedom" of science is – not untypically from our point of view – best understood from a legal perspective, as Pöschl has pointed out (Pöschl, 2017).

[8] In the history of philosophy, such problems were discussed particularly intensively using the example of the dualism of soul and body, e.g. by R. Descartes (Section 3.2).

[9] Hardly any other combination of mathematics and reality has such a strong impact on all of us as economic models and forecasts on taxes, economic activity, etc.

3 Didactical considerations on teaching such basic issues

For many reasons (and experiences), we advise against approaching the questions raised in Section 2 only through theory reading. Instead, we suggest starting with a motivating question, such as a historically relevant view on a connection between mathematics and reality in connection with a concrete mathematical task.

3.1 Pythagoreans: "All is number"

Let us start with a well-known and tested example. The Pythagoreans were a politically influential, i.e. not "only" philosophical school[10], of which many know the motto "All is number". Behind this motto was the desire to make human, social conditions more rational by – in today's words – mathematising them.

The secret symbol of the Pythagoreans was a regular pentagon, the analysis of which can be carried out in mathematics lessons and – not at all in the sense of the Pythagoreans – leads to irrational numbers[11] (Hischer, 1994). The method of taking alternate paths to determine the greatest common measure leads to the geometric interpretation of an irrational ratio of two lengths a and b: The taking of alternate paths ends exactly after a finite number of steps when the size ratio of a and b is rational. As long as $a \neq b$ in the case of $a > b$, the length a must be replaced by $a - b$ and $a > b$ must be checked again. If this is true, a is again replaced by $a - b$ and so on. Otherwise, $b - a$ takes the place of b. We see immediately that the side length and the length of the corresponding diagonal in the regular pentagon do not have a common measure, because the alternating path does not break off: in the inner, smaller regular pentagon created by the diagonals, we find the same (largest) common measure for side length and length of the diagonal as for the original. Therefore, these two lengths are incommensurable, their ratio is irrational. The Pythagoreans' motto "All is number" must therefore be extended to include such length ratios[12].

However, the lesson should not end there. If the motto "All is number" is formulated a little differently, e.g. like this: "Everything can be modelled mathematically", we can have the same exciting discussion in school that occupied the Greeks in the ancient: Is reality, the whole world outside of the science of mathematics, also mathematically structured? Can it be described (more or less completely) mathematically? Is this true at least for certain aspects of reality, such as all those that have to do with technology (e.g. Bardy, 2019), natural science (e.g. Ableitinger, 2010) and economics (e.g. Dorner, 2017), but not with the behaviour of individual people, for example in the area of personal relationships? These questions lead far beyond modelling in school, they are of such great importance for all of us that we want to give some space here in the form of a list to answer them.

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[10] Cf. the differentiated presentation in: https://plato.stanford.edu/entries/pythagoreanism/. Accessed: 21.05.2022

[11] For an introduction, see e.g. https://did.mat.uni-bayreuth.de/mmlu/goldenerschnitt/lu/anfang04.html or http://mathe-abakus.fraedrich.de/mathematik/pythagoras.html. Accessed: 21.05.2022

[12] The existence of irrational numbers can be proved more easily using the example of the diagonal of a square (root 2). However, a remarkable punch line, that the symbol of an influential group contains something that contradicts the motto of that very group. Crucially, however, the language of algebra was not available in ancient Greece. The geometric method of taking alternate paths is much easier to apply to a regular pentagon than to a square.

For us, the close connection between mathematics and the natural sciences is indisputable. Physics without mathematics is simply inconceivable, but also in biology, chemistry and medicine more and more modelling and – with particularly fascinating success for us – mathematical simulation is taking place. In the sense of B. Buchberger (RISC Linz), we simply attribute the computer science part of computer simulations to mathematics; behind every computer simulation there is (at least) a mathematical algorithm. Due to many contacts in the Johannes Kepler Symposium[13] with industrial mathematics or "technomathematics[14]", we know examples of the successful use of mathematics in the development of new technologies, for example: Energy saving in steel production, optimally planned chemical plant construction, better profiles for winter tyres, durable shape for springs in artificial teeth, planning of surgical interventions, bridge construction, heat protection for space shuttles, optimal farmland use in agriculture, safety check of cable car ropes, faster aircraft de-icing after landing and before the next take-off, minimised water consumption in washing machines and many others[15]. The ISTRON volumes[16] and the MUED materials[17] contain a whole range of teaching proposals in which applications of mathematics in science and technology have been worked up for school lessons.

The next item on our list are social and economic sciences. Obviously and publicly, mathematical models are used and argued with when it a matter of economics, i.e. in particular taxes or other incomes and expenditures of the public sector. A very good example for teaching is the dispute about the German motorway toll, because here the ministry and the ADAC came to very different calculation results about the expected toll revenues due to different assumptions in the modelling[18]. We know a lot about the use of mathematics in business administration from logistics (keyword "just-in-time" production and minimisation of inventory costs), statistics and advertising, controlling and financial mathematics, risk management in (investment) banks, etc.

After "armament research", "advertising" is the second keyword that perhaps entails concerns when talking about applications of mathematics. Let us dwell for a moment on the topic of advertising, because here it is easy to see how mathematisation is progressing. Long known (for decades) is the statistical research on advertising effectiveness, which examines, for example, the increase in sales through advertisements or radio or television commercials for a detergent. A considerable part of the advertising success of internet mail order companies is based on much more individualised targeting. A person's behaviour on the internet (search

[13] http://www.numa.uni-linz.ac.at/JKS/. Accessed: 21.05.2022

[14] https://www.mathematik.uni-kl.de/en/techno. Accessed: 21.05.2022

[15] Those who are not lucky enough to learn about new research personally from modelling professionals in industrial mathematics or technomathematics can find a lot of information on the internet, for example at scientific societies such as the Max Plank Society or the Frauenhofer Institutes. Many of these reports are deliberately written for a non-scientific audience – so they are also suitable as basic material for student presentations. Anyone who reads the reports with "mathematical eyes" will find traces of the application of mathematics everywhere. Incidentally, less is written about mathematics as a driving force of armament research.

[16] http://www.istron.mathematik.uni-wuerzburg.de/istron/index.html@p=1033.html. Accessed: 21.05.2022

[17] www.mued.de. Accessed: 21.05.2022

[18] https://www.wiwo.de/politik/deutschland/dobrindts-projekt-wie-hoch-sind-die-einnahmen/10109096-2.html. Accessed: 21.05.2022

queries, viewing offers in internet shops, etc.) is documented, analysed and translated into targeted advertising mails. From a person's behaviour on the internet, a cyber shadow of this person is calculated or modelled, which can be influenced individually better than through classic advertising spots. The example of "Cambridge Analytica" shows that this type of mathematical modelling can also be used to shape political opinion (Kaiser, 2019). For us, this reference is important at this point because it exemplifies how the limits of mathematisation shift from large numbers of people to individuals.

To us, the successful mathematics-based advertising on the internet seems annoying: we have to deal with it and our "real" needs or interests more than we would like, it also hits us more than roadside posters or TV commercials. There is only one positive thing we can gain from it: It shows us that an individualised approach has a better chance of attracting attention and interest than a generalised approach (cf. Eichler & Riemer, 2008, as a positive example).

On our list of areas that have not been successfully mathematised or, in our view, should not be mathematically modelled at all, we also find the term "aesthetics" alongside the individual. In art, since A. Dürer, attempts have been made to find out why pictures are "beautiful" or how it can be better to paint beautiful pictures with the help of mathematics. An example often used in the classroom is the "golden section" (e.g. Beutelspacher and Petri, 1996). In the meantime, there is far more mathematical knowledge about the "beautiful" design of objects in all areas of life summarized under the term "design" (cf. for example Jonak, 2012).

In music, Mozart already made an attempt at mathematical composition, his mathematical dice game, with which even people who know nothing about music and composition can compose music[19]. A more recent attempt in this direction is Tom Johnson's (permutation) Tango from 1984[20]. How closely the connection between music and mathematics can be seen is shown by the following quote from Heinz Götze: "[...] one could exaggerate and say that music and mathematics [...] are two aspects of one and the same thing. You can see this best when you open a score or a sheet of music. There is a mathematical coordinate system; in one the notes and the chords are indicated and in the other the time and the rhythm are indicated." (Götze and Wille, 1985, p. 1).

On the internet[21], it is currently possible to follow how mathematisation is progressing in many areas – e.g. on the way via "learning" neuronal networks.

One of the last items on our list is the behaviour of individuals in the area of personal relationships: Love, hate, loyalty, adoration or dislike, etc. If this can be modelled, we ask ourselves: does this, as with the Pythagoreans, also result in a political and social programme? Which one? A result of such discussions is open to us – in contrast to the analysis of the pentagon.

3.2 Leibniz: Determinate Monads

Newton (1643 – 1727) and Leibniz (1646 – 1716) are regarded as the founders of analysis (cf. Heuser, 1986b, p. 656 ff.); Newton gradually approached the infinite (limit values), Leibniz

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[19] https://www.ensembleresonanz.com/task/mozarts-musikalisches-wuerfelspiel/. Accessed: 21.05.2022

[20] http://www.editions75.com/FreeScores/TomJohnson(piano).pdf. Accessed: 21.05.2022

[21] For example http://www.medium.com. Accessed: 21.05.2022

uses his actual infinitely small "monads". Just to remind you: the limit thus goes back to Newton, the infinitely small dx on the other hand to Leibniz. It is not difficult to guess to which of the two philosophical starting points mentioned in Section 2.1 (Plato and Aristotle) the representatives of standard analysis and non-standard analysis (can) refer. An initial 'aha' effect can be achieved by answering the same question in class with the usual derivation and with Leibniz's monads (cf. Baumann and Kirski, 2016, p. 9 ff.): What is the derivative of a second-degree polynomial function? For this purpose, we consider for the unit parabola as graph of the function *f* with $f(x) = x^2$ an infinitesimally neighbouring point P_1 to the parabola point $P(x_0|x_0^2)$ with the coordinates $(x_0 + \alpha|(x_0 + \alpha)^2)$, where α is an infinitesimal number (monad, see next paragraph). It is smaller in magnitude than any positive number and not equal to zero. The horizontal distance between the two points is therefore α, which is one unit on the hyperreal number line increased by the infinite factor 1/α (which is a number larger in magnitude than any positive real number). The perpendicular distance is equal to $\alpha \cdot (2x_0 + \alpha)$, that is $2x_0 + \alpha$ units of the magnified straight line. The real part of the ratio of the perpendicular to the horizontal distance in units of the enlarged number line is therefore $2x_0$. This is the slope of the tangent line at point *P*. Both basic ideas (limits and monads) were the starting points for a long and successful development into standard analysis on the one hand and non-standard analysis on the other. Comparing the two can contribute to a better understanding of both types of analysis.

To make it exciting, we propose to talk about the monads in more detail in interdisciplinary cooperation with philosophy and theology. Leibniz defined and described "monads" in 90 paragraphs[22]. We focus here on the fact that they are "the true atoms of nature" (paragraph 3) and have "no extension, no shape and no possible divisibility" (paragraph 3). Monads can only begin by creation (paragraph 6). They are windowless, i.e. no means can "explain how a monad can be influenced or changed in its interior by another created thing" (paragraph 7).

If we ask about the meaning of monads and their activities (e.g. as bodies composed of monads), we learn from Leibniz: "Therefore the ultimate ground of things must lie in a necessary substance, in which, as in the source, the particular of changes is only essentially contained, and this very substance we call God." (paragraph 38). God in his infinite wisdom created the monads and in the course of creation determined their destiny for all time – we therefore live in a determined world.

Can we believe in this model of reality? Does it convince us? We can (perhaps with some additional reading) recognise and admire how the monad model solved the back then almost insoluble problem of the dualism of soul and body that Descartes, for example, used with so much success[23]. Moreover, we should discuss an input from the theological direction: how is this fully deterministic model of monads compatible with the problem of theodicy? If humans are also made up of such monads, how can they make decisions with free will? Furthermore, we should mention here (again closer to mathematics) the phenomenon of sensitivity to the

[22] https://www.hermetik-international.com/mediathek/historische-schriften-der-mystik/gottfried-wilhelm-leibniz-die-monadologie/. Accessed: 21.05.2022

[23] An immortal soul is in a mortal and perishable body. The model that can be only mentioned here explains many things quite vividly, but not the connection between soul and body: cf. https://www.argumentarium.ch/philosophie/leib-seele/77-dualismus. Accessed: 21.05.2022

initial conditions of some (chaotic) systems, which relativize the determinism of such systems (e.g. Raith, 2009, Bigalke, 1996 or Dormayer, 1991). The description also influences the calculated behaviour of such systems: discrete or continuous in the case of logistic growth (e.g. Ableitinger, 2010, Section 2.1.4).

A deterministic philosophical basic position is not compatible with our didactic concern of "reality-based mathematics teaching". In the process of modelling, decisions have to be made again and again (cf. Section 5): Is the model developed so far sufficient? Do we need additional data? Should we model further aspects? Students should learn to make such decisions on their own responsibility and to draw the appropriate consequences. However, the binding nature of one's own decisions for further work quickly will sink to zero if the students believe that everything has already been decided and predetermined anyway. Then it does not matter what they decide and do. With this conviction, it is obvious to make things easy for oneself and do nothing. The motivation to behave “good” sinks to zero.

3.3 Hume: Reality is sensation including interpretation in one's own head

With David Hume (1711 – 1776), we have chosen another very important philosopher who formulated a sceptical view of reality. This view is very disturbing for most people with a usual and rather naive view of reality, and for this very reason it is appealing for teaching on the topic of "reality".

"The problem of the external world consists in the philosophical question whether the external things around us exist independently and differently from our perceptions. Hume dealt with this problem in his Treatise on Human Nature, among others. He stated that the belief in the existence of the external world could not be supported by rational reasons. According to his basic sensualist thesis, the senses are the only source of our knowledge of the external world, and these only provide us with perceptions, but there exists not the slightest indication that our views or interpretations are caused by something outside of themselves.

‘The function of the senses is probably unsuitable in order to be able to derive from it the idea that things are continuously present long after they have disappeared from our senses. We arrive at contradictory statements when we assert this. [...] The senses deliver only single perceptions without the slightest hint of something outside and different from us'.’

Nevertheless, man cannot avoid believing in the existence of the external world. According to Hume, nature has not left this to [man's] choice. He raised the question of the reasons for this strong belief.

'Now, if I ascribe a real and corporeal existence to these perceptions, something happens in consciousness that is difficult to explain, but which I will try to do.'"[24]

Those who look at the same object again and again may experience the permanence of the same sensory impression of this object and may thus be tempted to believe in the existence

[24] https://de.wiki-pedia.org/wiki/David_Hume. Accessed: 21.05.2022. See: Hume, 1896, Book One, Part IV, Section II.

of the object independently of themselves. This is psychologically understandable and exculpatory, but not a proof of existence. An essential sharpening of this position took place through nihilism (or solipsism): According to this, no reality exists except the one in one's own head. A successful combination of nihilism and mathematics didactics is not known to us and even difficult to imagine.

We have tried to relate Hume's philosophical position to a mathematical problem that is understandable in school and propose to treat it in this way: On the occasion of the anniversary "50 years of the moon landing", a fictitious school class (or an agency doing opinion polls), which is to be reported at the beginning of the lesson, made a survey. More than 1000 people were asked: "How far is the moon actually from the earth?"

After careful evaluation, the result was approx. "4800 km". As is usual and necessary for a good evaluation, first the meaningless answers ("My heart is only one step away from the moon"), and the few obviously only jokingly mentioned giant numbers were sorted out and then the mean and median were determined according to the rules of art.

Have we understood Hume correctly if we conclude from this result that the moon is actually about 4800 km away in reality? How does the answer turn out if the survey does not concern astronomical "facts" but political assessments (does person x hold the right view on topic y?)?

The problem of the second child can also be discussed in relation to Hume's conception of reality: We know that a family has two children and at least one girl. What is the probability that this family has two girls? Depending on the view and interpretation, this can result in the value 1/2 or 1/3 (Götz and Humenberger, 2008). The decisive factor here is how the information about the sex of a child comes about (Götz and Humenberger, 2008, p. 52). For example, the assessment is based on a sensory impression (a girl belonging to the family was encountered by chance in the stairwell), the other child does not become known. If, on the other hand, the question "Is there a girl in the family?" is answered with "Yes", the probability sought is to be evaluated differently (Götz and Humenberger, 2008, p. 55).

In the taxi problem, an evaluation is also made based on an observation: In a city there are N taxis with the numbers 1, . . ., N, which can be read from the roadside. A passer-by stands for a certain time on a busy road and notes down the numbers x_1, . . ., x_k of the passing taxis. Assume that $1 \leq x_1 < \ldots < x_k$ and that the passer-by counts a taxi passing by several times only once. Assuming that all taxis are in operation during the observation period, we estimate the number N of all taxis. One possibility is to estimate the length of the unobserved gap $\{x_k + 1, \ldots, N\}$ above x_k by the mean length of the preceding gaps. Then we get $\frac{(x_1-1)+(x_2-x_1-1)+\cdots+(x_k-x_{k-1}-1)}{k} = \frac{x_k-k}{k}$. This leads to the estimator $x_k + \frac{x_k-k}{k}$ (Krengel, 1991, p. 65).

In our opinion, the examples themselves fit very well with Hume's view: the (sensory) impression leads to the evaluation or assessment of the situation. However, this is not enough. In the first example, the way in which the sense impression came about plays the decisive role. In the second example, on the other hand, the choice of the estimator is decisive. In both cases, therefore, both a sensory impression and a theory are necessary, not to grasp "reality", but to arrive at assessments of different possibilities.

How do we deal with such fundamental relativisation in matters of reality? What can we still really rely on when the naïve certainty of sensory impressions or faith in God or in the truth of the results of scientific research has left us? In the realm of everyday life and technology, we are left to assume that the things we use and need (from the household to public transport) are, if not absolute, at least provisionally real and usable. We (hopefully not!) fall into a metaphysical hole when we use the coffee machine or watch a film in the cinema. The certainty about the actual situation in Syria (or other countries at war) or the current state of interests in the matter of "Brexit", which can be achieved even with the greatest effort, seems much less to us.

To sum up, like most people, we get along quite well in everyday life if we think this is "reality" and behave as if it is really reality, what we can grasp with our senses. It helps that we grow up and are educated or socialised in social contexts. It becomes interesting – and sometimes critical – when we have contact with foreign cultures, for example, in which other common views prevail, or when we enter the realm of science. To put it very simply, in everyday life we can usually simply ignore what physics tells us about relativity or quanta. However, if we ourselves want to pursue a science like "didactics of mathematics", we have to take a close look at the conceptual foundation, the (also empirical) conclusions on which we want to build our theories. Otherwise, we risk justified criticism.

With regard to reality-based mathematics teaching, we have the impression that we can address simple examples from the learners' everyday life (household, everyday technology, financial examples for lower secondary school, etc.) with little risk. If we deal with or develop models, which are more complex, e.g. from economics, ecology, or the theory of modelling, we have to take into account much more precisely that there is not one and only one simply reality that can be modelled: see Section 4.3.

4 "Modelling" – a proposal for an expanded understanding of the term

Whoever talks about "modelling" in German-language mathematics didactics today usually uses a modelling cycle according to Werner Blum or another ISTRON member (cf. Blum & Leiss, 2005) to explain it. In these cycles, the movement is from reality to mathematics and back to reality (which is usually taken as a fixed given without comment), whereby more or less many stations are explicitly mentioned depending on the author (e.g. the use of a computer for calculations is not mentioned by all, but taken for granted). For reasons of space alone, we cannot go into the different variants here. However, the previous two sections show that this trust in a fixed reality can at least be questioned scientifically (philosophically). At the end of the last Section, we have deliberately stated that our brief excursion into philosophy has already made it clear that such an understanding of "reality" as the basis of scientific work can be seen as too narrow.

Furthermore, "circle" is a misleading metaphor for a desirable learning process. Of course, those who learn to walk in circles also learn. However, those who only run in circles will not get far in the end. Instead, we suggest an opening spiral as a designation and guiding principle (Maaß, 2015, p. 202, cf. Section 5).

Another type of criticism aims at the fact that simplified modelling circuits with fixed reality are not appropriate insofar as corresponding tasks are not about "reality" at all, an authentic reference to the students' lifeworld, but about a "mathematics teaching artificial reality" as it is known from common text tasks (especially in older textbooks): "The (open top) filling funnel of a cement silo consists of a rotating cylinder, to which a rotating cone (with negligible filling opening) is connected at the bottom. [...]" (Götz and Reichel, 2011, p. 149). We therefore propose to speak of modelling and reality reference in didactics and teaching only when it is really about an authentic reference to the students' lifeworld. Have you noticed how, even with this attempt at delimitation, we are now struggling with the problem we are concerned with in this article: What is the "actual", "authentic" reality, the students' lifeworld, and why is the "mathematics teaching reality" not part of it? It is obviously important not to take simply "reality" for granted as a fixed quantity.

At the centre of reality-based mathematics teaching, which is described in many publications, for example by ISTRON members, and is now required in some form in all curricula and competence catalogues for standards and central leaving examinations in German-speaking countries, is "modelling": Students should acquire modelling competences (Siller and Greefrath, 2015). This would be sometimes discussed as if it was a completely new requirement for them, as if something had to be learned in addition to all the important material (the mathematics taught so far). Such a view of modelling mobilises defends instead of motivating. That is why we remind you in this article that we, just like all students, model as a matter of course in everyday life, i.e. we build and use models. We claim: The special thing about mathematical models is that they can help to improve the quality of these activities, for example, to make predictions that are more accurate or to systematise something more selectively. Therefore, our central message to learners and teachers is: those who can use the power of mathematics in modelling will understand and influence the world better.

The following remarks are based on Maaß (2015) and Maaß et al. (2018), from whom we have summarised and reworked some arguments for this article. To start with, we use a graphic that shows a different modelling circle than the one usually used in the ISTRON environment, namely a much more general one. The diagram in Figure 1 shows people, reality and models in a respective (dialectical) interrelation.

The central message of this graphic is that we all use models as a matter of course in our dealings with what we understand as reality. The second central message is the respective interaction between people, reality and model. People create models (Section 4.1) and change reality – also with these models; reality obviously has an influence on people and models with which to describe and change it (Section 4.2). Models (e.g. for calculating income tax) influence people and (at least the) reality they perceive in this context (Section 4.3).

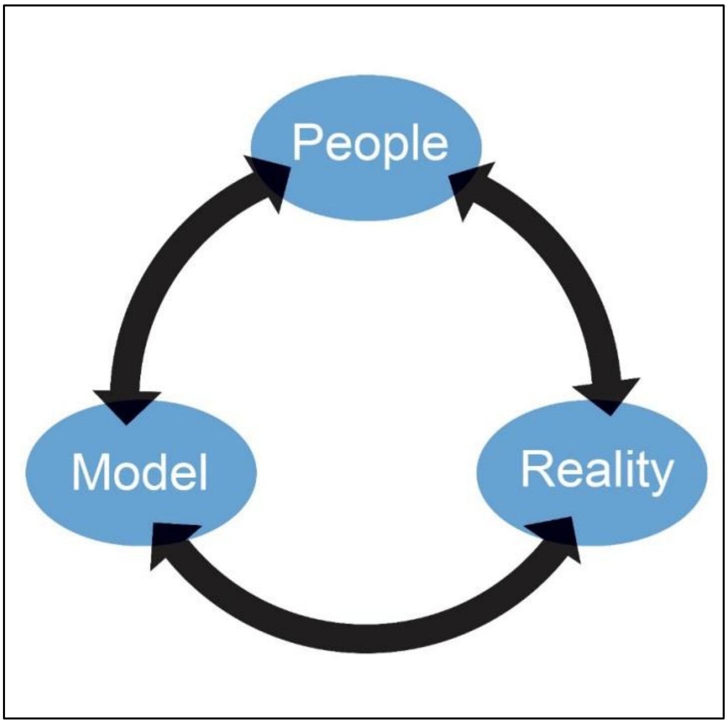

Figure 1: Modelling Circle (see Maaß et al., 2018, p. 190)

4.1 Starting point: Models are a means of cognition in everyday life

To explain this basic form in Figure 1 (a detailed version follows in Section 5.1), we first discuss a few simple examples, also to show that there is not just one term "model" that means exactly the same thing in all uses for all users. In view of this diversity, we come at best to a less exact description than desired – for example, that a model is a simplified version of a real complexity.

When someone goes shopping, he (or she) uses models of objects, people and behaviours to plan a route or write a shopping list, for example. Someone who wants to visit every day the baker around the corner does not think long about the way there. It is present in the mind – as a model, e.g. as a section of a city map or as a sequence of path segments. Of course, no one has the path itself (as matter, as it is, the whole pavement, the adjacent houses with tons of bricks, etc.) in the head, but an idea of it or a memory of how the path was the last time they went to the bakery. The difference between a model of the path, in which parts of the path are combined with memories (for example: from the front door turn left to the corner, then cross the street and continue to the entrance of the shop), and a map of the city is also a difference in the degree of mathematisation of the model. If you also want to try out whether the new smartphone with GPS shows you the same way, you will experience the function of an extremely highly developed mathematical model: for example, many solutions to classical problems of navigation, mathematical modelling of the earth's surface, satellite control and communication with them. There is a lot of mathematics in the GPS, the smartphone, etc. (cf. Greefrath and Riemer, 2013, Maaß and Spiegl, 2020). Both are fine examples of mathematical technology that functions as a black box, even if a user does not understand the mathematics used or has no idea that mathematics is being used here at all (cf. relevance paradox)!

We extend the shopping example to remind us of another aspect of modelling that we are very familiar with in everyday life. If you want to go to a newly built shopping centre on the outskirts of the city for the first time, you might need a model that is more recognisable as such, a city map or a public transport map to plan the trip. You also might ask someone who has already been there and knows the way for directions (i.e. a model of the way!). In other words, without much theoretical thinking about modelling, someone tries to improve his or her model of the way so that he or she actually reaches the desired destination. In doing so, most people proceed pragmatically. The (expected or experienced) practice helps to decide when a solution is good enough. Who only sets off when it has been mathematically proven that the chosen route is optimal? We emphasise the pragmatic approach to assessing the quality of models because it can also be helpful as a basis for decision-making in teaching projects.

To conclude the shopping example, we would like to point out how naturally we use the term "model" in many different ways in everyday life. If you want to buy a new pair of trousers, you may have a model in mind, a picture from an advertisement or a – human – model who wore these very trousers at a demonstration (fashion show) that was shown afterwards on advertising television. Other examples are toy models such as a model railway, a model aeroplane or professional models such as that of a planned house or shopping centre shown by the team of architects to the client.

4.2 Models as a means of cognition

In Section 4.1 we recalled that models are a means of cognition in everyday life as well. We will now sketch a few steps into epistemology to justify this thesis, but we do not want to deter with a discussion in philosophical jargon, but deliberately explain what is at stake in a didactic way. For the sake of simplicity (and because it is closest to mathematical cognition), we will limit ourselves here to considerations of optical perceptions.

The eye – when it is healthy – provides us with optical impressions in light, more precisely a sequence of light/dark and colour impulses that are converted by cells in the back of the eye into signals that the visual centre in the brain can interpret. This interpretation of optical stimuli by the brain is the crucial point! An electronic camera records optical information (e.g. 24 images per second) and stores it – uncommented! – on a storage medium, such as a hard disk. As it is well known, our brain, unlike the electronic camera, differentiates which images (or optical information) it takes note of at all, which it reacts to and which it simply ignores. The brain works as a filter, we concentrate – perhaps – on the important, the significant, the directional of all that we see (cf. learning process-oriented didactics, e.g. Schukajlow-Wasjutinski, 2010, Section 5.1). A very central role is played by the recognition of familiar things (experience), such as objects like a book or a car, or people we already know. When we see a car, for example, the brain is supported by the fact that it has stored a general pattern of a car, a model of a car. A car has certain characteristics (such as four wheels, a certain size, number of seats, windows, ...) so that the brain does not compare pixel by pixel like a (badly programmed) computer does when recognising objects, but it analyses based on characteristic features. From the multitude of cars seen, the brain creates something like a typical car or a model of a car, usually differentiated with subcategories such as sports car, estate car, SUV etc. This formation of patterns or models is an essential learning process, it helps to perceive quickly and efficiently. Someone standing in the middle of the road and observing a strange thing coming towards them for the first time in their life has very little time to learn about cars, road traffic and traffic rules (such as "Be careful when crossing roads as a pedestrian!"). On the other hand, people who have been driving routinely for years often leave it to their brain (the non-conscious part) to recognise the important (traffic) signals and movements of other road users and to react appropriately to them (for example, by braking before a red light) – and they concentrate on the music from the car radio or a telephone conversation while driving.

Of course, the brain does not only store patterns or models of cars (if you see a report from a car show, you will recognise without much thought whether a new car is conceptualised as a sports car or a station wagon, for example), but of everything: living beings, technical objects, faces, ... In this way, we indicate in what – comprehensive – sense existing models determine our perception of reality (or of what we think it is). When we go for a walk and see plants and animals, we recognise deciduous trees, hedges, flowers, birds, insects etc. with the help of what we already know about such plants and animals. In other words: We have biological or horticultural knowledge (which can be more or less correct or wrong) to assign the plant or living being before our eyes to a type (of which we in turn have a model, such as a more or less correct definition, in our heads): this is a fir, this is a beech, this is a rose, that is an – obviously diseased – oak (we see many dead branches), etc.

In everyday life, another type of model plays an important role: When we watch children playing with a ball, more or less well-understood scientific models help us or, again, we are supported by experience in order to have an expectation: Will this ball go into the goal? Will the running child reach the ball before it falls into the stream or rolls onto the street? This is not about comparing things seen with patterns (or models stored in the brain), but about ideas of how objects or living beings typically move, e.g. how they fall. This kind of model can obviously become more precise through explicit mathematisation!

To conclude this little excursion into philosophy, let us recall that our argumentation on cognition and models has long been known in philosophy, so it has not been reinvented here. Sir Karl Popper, for example, wrote in his "Logic of Research" published in 1934: "Our everyday language is full of theories; observation is always observation in the light of theories." (p. 31). Theories, in the sense used here, are also models of objects, behaviours, organisations, etc.

4.3 Interaction: Models influence people

Why is there also an arrow in the opposite direction in the diagram in Figure 1, i.e. from the models to the people? Obviously, models have an effect on people; they influence the possibility of perceiving something. If you have a model-like idea of how a car, a PC or a virus works, you can use this idea to make better use of a car or a PC or to protect yourself from a virus. In medicine in particular, there are many examples of how model ideas of organs, bacteria or genes contribute to advances in diagnosis and therapy ("functional model"[25], Morgenstern, 2011). Such models can also play a role in making people behave in a "healthier" way.

[25] E.g. for vocal cords: https://www.sammlungen.hu-berlin.de/objekte/lautarchiv/9313/. Accessed 21.05.2022

Models have also a significant impact on all of us in everyday life and society. As examples, we mention economic models that are used to justify tax increases or tax cuts (cf. Section 2.5) or climate models that are used to refer to changes caused by human energy use (cf. as an early example Volk, 2003).

The keyword "human energy use" reminds us that although the word "people" appears in the graphic, it is not meant to claim that all people are the same, have the same perceptions of reality, interests in change and similar opportunities to influence. Someone who lives in Europe or North America in a house with heating and air conditioning uses significantly more energy than someone who lives on the street in Africa or Central America. Someone who speculates with food on the stock exchange has a completely different access to grain than someone who cannot feed his family because of high grain prices. Someone who cycles on holiday needs much less oil than someone who flies halfway around the world on holiday. Incidentally, a large number of teaching materials, for example in the MUED, shows how closely ecology, energy use and mathematics are linked.

Equally different for different people are their approaches to reality or their perception of it; this in turn is also related to which models of aspects of reality they have in their heads and they can use. We are aware that the word reality is used here in a philosophically rather naïve way.

In other words, the graph itself in Figure 1 is a very simplified model of human interaction with reality, in which modelling always plays a role.

Finally, with regard to Figure 1, it should be noted that there is obviously a dialectical interaction between reality and the human being. On the one hand, social, economic, societal, ecological and personal reality provides a framework for each person to move within. On the other hand, people influence this framework and thus reality by trying to recognise and improve their situation.

5 "Modelling" is at the centre of reality-based mathematics teaching – a proposal for an extended understanding of the term[26]

5.1 On the role of mathematics in modelling

In addition, where is the mathematics? Modelling with mathematical methods is a matter of course in research. Research reports from natural and social science, as well as those from other fields of science, usually contain mathematical formulae (some people call these formulae "laws" in the hope that nature and society may adhere to the "laws of nature") and references to mathematics used as justifications for the correctness of the results or the correctness of the research methods. Our central thesis for the role of mathematics in modelling is that its use improves the quality of all models in which regularities can be described with formulae, equations, or systems of equations and thus predicted. The history of the natural sciences is so rich in evidence for the thesis that we need only recall astronomy and navigation as well as mechanics and analysis here to justify it ("I maintain, however, that in every particular natural science only so much actual science can be found as mathematics

[26] Cf. Maaß and Grafenhofer, 2019, p. 4 ff.

is to be found in it" I. Kant wrote in 1786: Kant (2014), p. 6). Anyone who doubts the thesis is invited to search the internet under the keyword "industrial mathematics" or "technomathematics". The search results reinforce the impression that mathematical modelling opens up a more precise and deeper insight and possibility for change for a large amount of aspects of reality. This is especially true for scientific, technical and economic topics; however, when individual human behaviour or psychological factors are to be modelled, the limits of meaningful mathematical modelling quickly become apparent.

In most people's everyday lives, explicit modelling with mathematical methods is rather rare. We fear that many people therefore miss out on an opportunity for more rational decision-making or a better understanding of social, economic (cf. Dorner, 2017, Section 1) and ecological developments and therefore we plead for reality-based mathematics education, too.

In order to point out the possible role of mathematics in modelling more clearly, we expand the graphic in the area "model": Figure 2. The work on the mathematical model is the focus of the lower part of the graphic here, which looks into the area "model" with a magnifying glass, as it were. With the aim of creating a mathematical model, the required data together with its structure are filtered out of the model for those aspects of reality that humans have set themselves as goal of recognising or changing. It is important to note that there is not simply a TOP-DOWN structure here in which a planning person models and mathematises reality according to his or her will – even if many people would like to be able to do so. In fact, it is a matter of manifold interactions. Even in the selection of aspects of reality to be considered, one criterion is which aspects can be meaningfully described mathematically at all. In other words: If something is to be mathematically modelled in a school class, the self-assessment "What can we do at all?" is naturally also part of the decision. Technical or scientific aspects are much more likely to be modelled from this perspective. Emotions and other psychological aspects are therefore usually ignored, also social relationships or esoteric matters. At first glance, this makes a lot of sense: if, for example, stopping distances of cars are modelled in order to understand hazards or congestions better, the colour or design of the cars is quite rightly ignored at first. If, after some refinements of the technically oriented modelling, one notices that road users behave differently in the observed reality on the roads than in the model, it may make sense to model their emotional situation as well. Does stress or time pressure lead to lower safety margins? However, modelling such relationships can quickly become mathematically very demanding!

Figure 2 attempts to take into account the dynamics of modelling by several runs that are visually drawn in (cf. the small circle in the mathematical modelling part below right). Mathematical modelling, as indicated below in Figure 2, is actually "only" a special component of modelling, a specific and in some respects particularly effective method of modelling.

When the mathematical work in the narrower sense is done for the first mathematical modelling, this often results in wishes for the modelling and attempts to improve it through better data, more information about the structure of the data or targeting only certain aspects of reality. More appropriate mathematical modelling is also often sought. There are no fixed rules for after how many such runs the result meets the wishes for further work on the subject should be discontinued due to lack of time, lack of further possibilities for model improvement or other reasons. In this context, it should be explicitly mentioned that technology continues to play an important role: The possibilities and limits of the use of mathematical software has a great influence on the selection of topics, aspects of modelling and on the chances of arriving at an adequate result. Many topics can only be addressed today in research, industry and schools because technology provides sufficient support.

The boxes for "better model or new aims" and "changed reality" are intended to remind us that (mathematical) modelling can and should have consequences, which in turn can and should have repercussions for people. One possible repercussion of, for example, exploratory modelling is often getting new insights and receiving questions that lead to new modelling.

Following a suggestion by I. Grafenhofer (Koblenz), which we are happy to take up, we would like to point out here explicitly that the lower (blue) part of the diagram, in which specific reference is made to mathematical modelling, can also be replaced or supplemented by a (e.g. green) part in which biological, chemical, physical, economic etc. modelling is presented. Clearly, modelling succeeds most convincingly when knowledge from all relevant sources is included. An example of this are models of climate change on our planet.

Figure 3 is meant to visualise that this is an open process leading to an uncertain end: The starting point is again WE (W_i), i.e. people who want to understand or change something, model for this purpose and thus bring about consequences for reality (R_i). These consequences or other motivations lead to renewed efforts, to new and hopefully better models, which in turn have an impact on reality. When and how the effort to improve cognition or reality (normative models!) ends is in principle as open at the beginning as the final evaluation of the process: Has an improvement actually been achieved (or from whose point of view?)?

At the beginning, there is a decision: the desire to recognise something, to understand something better, to change, to achieve faster, to control with little(er) effort or to be able to use resources as efficiently as possible. All this and much more can serve as motivation. After an aspect of reality has been selected that can and should be examined more closely with mathematical methods, data and regularities are sought with which these data are linked. At the beginning, usually not all the necessary data are available and it is not always clear in which way the data can be structured with the help of the mathematical toolbox or can be represented or described by a mathematical connection.

The start in the first run is therefore often based on estimates and very simple mathematisations. At first, (mathematical) models are created of which it is known that they by no means consider all parameters. Nevertheless, typical consequences are questions for more precise or additional data, for more complex mathematical tools or the desire to specify the question or the objective (cf. Borovcnik et al., 2018). If one or more of these questions have been thought through, a second experiment will be carried out; its result must again be interpreted.

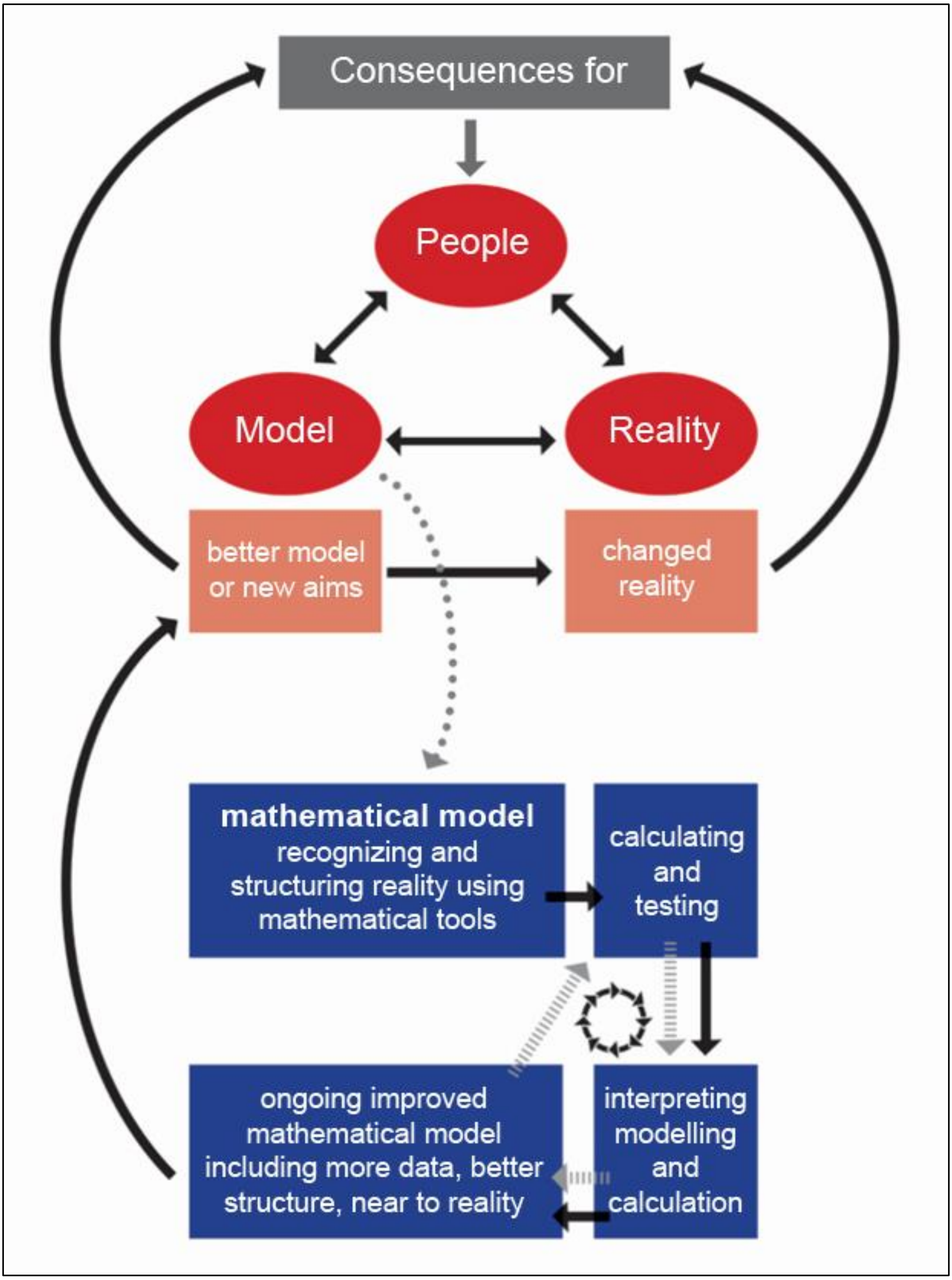

Figure 2: Extended modelling cycle (see Maaß et al., 2018, p. 194)

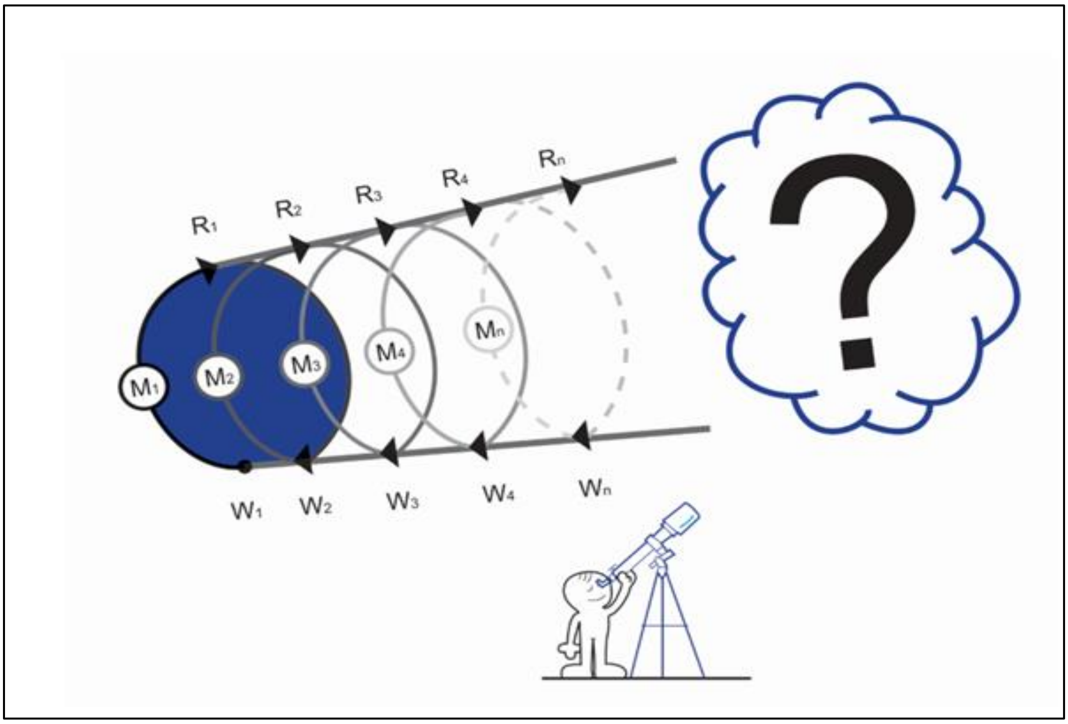

Figure 3: Modelling with perspective (see Maaß et al., 2018, p. 196)

Analogous to the first pass, the acting persons have to decide whether and how to proceed. This is often a subjective decision that depends on time and skills, motivation and many other factors, but is not objectively predetermined by mathematics (such as when proving a theorem). After a few passes, the acting persons have usually found out with sufficient accuracy what they wanted to know or realise that they are not making any progress despite of all their efforts because, for example, better data are not accessible, mathematical representations exceed cognitive abilities or they simply have the impression that further efforts are no longer worthwhile because the amount of work is becoming too huge.

5.2 Responsibility for the consequences of modelling in reality[27]

At this point, we particularly point out that the activity of (mathematical) modelling involves more responsibility for the results than it is usually the case with arithmetic tasks. When an algorithm is practised for the next test, it only matters whether the result is correct, but not what it means for reality or what consequences (may) follow for the people concerned from the application of this result. Even if a school class plans to sell homemade cakes at the parents' evening in order to better finance the next class trip, the experience of successful modelling (for example, the considerations about the amount of cake that can be sold were just as correct as the planning of the shopping and the baking itself) can be pleasing. However, it may not be so successful – and already the question of responsibility is raised.

It is much more dramatic in the case of professional modelling, i.e. the result of the work of mathematicians: if the load-bearing capacity of a bridge or a load-bearing cable in a cable car

[27] Cf. Maaß and Grafenhofer, 2019, p. 6.

was modelled and calculated incorrectly, the consequences can be catastrophic – just as in the case of an economic model oriented towards one-sided interests, which leads to laws (for taxes, environmental pollution or subsidies) that have dire consequences in the long term. It is not new and unknown to the pupils that their actions in the real world (can) have consequences for which they have to take responsibility (keyword: ball game and glass pane in the neighbour's house). What is new for many of them is only that in reality-based mathematics lessons the results of their efforts (can) have consequences as in real life outside the classroom.

6 Reflections on didactics

6.1 Three theses

Before we make or repeat some suggestions on the didactics of reality reference in mathematics education, we summarise our view of the reasons for this in some theses, which are substantiated both in the previous empirical research results of ISTRON members on reality-based mathematics education (cf. for example Maaß, 2004) and in our and international empirical research on the relationship of adults to mathematics (cf. in summary Maaß, 1994, as well as many contributions in the ALM Journal[28]).

Thesis 1: Reality reference can provide very convincing and motivating answers to the question of the meaning of mathematics teaching.

Somebody who has calculated the optimal tariff (provider) for his or her mobile phone in class and actually saves money consequently will certainly no longer ask: "Why should I learn this?" (cf. Maaß, 2002). Many topics from the field of energy and environment can be similarly convincing if they are related to the learners' horizon of experience. Therefore, it is better to think about one's own consumption behaviour than that of all the inhabitants of the USA. But even "adult topics" such as financing one's own home (cf. Dorner, 2017) or the quickest way to the nearest hospital (cf. Lutz-Westphal, 2007) in case of a heart attack can sometimes become more interesting for the learners through a good frame story: We help the aunt, who is not so good at maths, with the planning. She has received the following three offers. Which one should she accept? Alternatively, for the second example: The neighbour was the doctor’s; he has a problem with his heart. What can you do if he quickly has to go to hospital?

Thesis 2: Connections between mathematics and reality have fascinated some people at all times to such an extent that they have spent their entire lives thinking about them.

We owe many an important insight and many open questions to these people, who inspire us to think further (cf. Section 3). If we help to bring more students into contact with aspects of this topic at school, we can hope that some of them will also engage intensively with it beyond school. We are very curious to see how the discussion of the topic expands the hoped-for new impulses.

Thesis 3: Reality reference expands the image of mathematics with which learners leave school.

[28] http://www.alm-online.net/. Accessed 21.05.2022

If the operational aspect of mathematics, the training of solving different types of problems, dominates too much in school lessons (which has been and still is criticised again and again), adults will look back on their school years with a very one-sided and – from our point of view – wrong image of mathematics. This is regrettable for many reasons. Firstly, such an image prevents many people from using mathematics for their purposes in everyday life. Far too few people can actually recalculate financing, choose favourable tariffs based on their own calculations, calculate handicraft things in their own home in advance (geometry and finances!) and, on the other hand, far too many falls for manipulative statistics (cf. Krämer, 2015), interest-driven mathematical modelling of economic things, etc. Further, such people are missing when the economic location is discussed. Who can believe that mathematics is the basis and the secret of success of all new technologies and who can enthusiastically devote himself or herself to the creative and constructive aspects of mathematics if all he or she knows from school is that mathematics consists solely of endless collections of meaningless (arithmetic) tasks?

We explicitly point out here that solving arithmetic tasks, practising algorithms as before and teaching to understand the connections between mathematics and reality are only two of many aspects that make up a reasonably rounded or complete picture of mathematics. Arguing and proving (Bürger, 1979; Sattlberger and Götz, 2007), historical developments, connections ("Linkages", see e.g. the "Mathe vernetzt" volumes at MUED) of mathematics to art, music, natural sciences and computer science, linguistics and technology, philosophy and theology, sport and medicine are only a few further keywords in this regard.

Finally, we mention a very selfish reason. People who have a bad image of mathematics can also decide about mathematics as politicians. We are quite afraid that one day we will be like our colleagues in biology, who are prescribed "intelligent design" by law[29]. Are these exaggerated fears? We recall the enthusiasm with which the headline "Seven years is enough!" was taken up and disseminated in the media with regard to mathematics education (cf. Heymann, 2013).

6.2 Counterarguments

Whenever it is desired or demanded that something new or something already known be covered a little more in class, there is a typical defensive discussion. The core argument against this is clear, simple and very convincing for most people: we already have too much material – more is not possible. What value have all the beautiful – and to us convincing – arguments for the topic "Connections between mathematics and reality" if they bounce off this blocking wall? Nothing. Therefore, we have to deal at least theoretical with the central counter-argument.

First, we would like to point out that in all curricula known to us in the German-speaking countries mathematics teaching in our sense is required. There is nowhere a state requirement to limit teaching to "training-to-the-test". However, we see the problem that many teachers interpret material catalogues from curricula or competence catalogues very comprehensively, i.e. teach more tasks or aspects of a material area than it would "actually"

[29] https://www.findlaw.com/education/curriculum-standards-school-funding/creation-evolution-and-intelligent-design-in-public-schools.html. Accessed 21.05.2022

be necessary to meet the official requirements. However, the freedom to interpret the existing subject matter catalogues is the opportunity to address other perspectives on the same subject matter (mathematics) in the classroom. In other words, the material is not "objectively" (by the employer) overwhelming, but only subjectively because of the teacher's interpretation. Therefore, there is room for manoeuvre that can be used. If we think, for example, of the construction of triangles, then this leeway is very large from the specification of the three side lengths to the specification "A triangle is to be constructed of which the sum s of the sides *a* and *b* (with $\overline{BC} = a$ and $\overline{AC} = b$), the size of the angle *ACB* and the length h_a of the height falling from *A* onto the straight line through *B* and *C* are given: s = 7 cm, h_a = 4 cm and γ = 100°"[30].

Maybe the motivation is lacking rather than the opportunity. Then let us make a motivating argument! Anyone who wants to know thoroughly empirically or simply observing themselves and other people who have learned mathematics, what the relationship between mathematics and adults is like, will come across to two facts:

1. Mathematics skills are very low. What is not used in work or everyday life is quickly forgotten.

2. Attitudes towards mathematics are often negative. If the school subject of mathematics plays a role in films or books, in dreams or memories, it is usually a negative one.

If this is true (and there is no reasonable doubt - cf. in summary Maaß (1994) or the international contributions of Adults Learning Mathematics), then it is not a disadvantage if some of the usual material is dropped so that the pressure is reduced and instead something new, positive, motivating takes place in mathematics lessons, for example on the topic of "connections between mathematics and reality".

In our experience, such special teaching sequences are better remembered than "normal" ones and help to create positive impressions of mathematics teaching. In short, anyone who introduces something along these lines into mathematics lessons is on the right track to improving the image of mathematics as a whole!

6.3 Setting the course

Let us start with a small example: Andrea has read a book that she liked very much. At the end of the book, she finds a note saying that there are two sequels to the book. The two books cost 7.90 euros and 8.90 euros. The typical task is now: How much do both books cost together?

Instead, we suggest asking a little more openly: What can Andrea do to read the two books? Can she borrow the books? Maybe other students have all three books and are willing to borrow them? Maybe Andrea will learn on this occasion that a library will lend her these books. In this case, Andrea has learned something for her life that is much more important than 7.9 + 8.9 = 16.8 (euros). Do we hear the objection that this is not mathematics and does not belong in mathematics lessons? Reality-based mathematics lessons are about actually

[30] https://www.olympiade-mathematik.de/pdf/klassen_al/klasse_07_al.pdf (Aufgabe 020736). Accessed 21.05.2022

solving real problems, not just calculating! When we look for second-hand copies (on the internet), we can ask questions about the (percentage) reduction in price compared to the new price.

This small example should emphatically point out that we think it is very important in such mathematics lessons not to work merely with fake news, but to make a comprehensible attempt to solve actually real problems. Questions from practice should not only be a pretext for or introduction to typical textbook tasks.

In this sense, it sets the course for a different understanding of commitment in mathematics education.

6.4 More concrete means more complex

Mathematics is particularly distinguished by its power of abstraction compared to other sciences. If we add the natural numbers 2 and 2, we always get 4, no matter what unit we use or what objects we have in mind. If, on the other hand, we ask the question in class how far we have walked if we first walk 2 km and then another 2 km, the correct answer is "4 km" only at first glance. If we look at real distances, we have probably walked 3.92 km or 4.15 km, for example. Moreover, when recalculating, we have to discuss for a very long time whether and how we take into account differences in height (a kerb, a slope) (cf. Maaß, 2011). Then we are immediately faced with quite difficult and exciting questions of modelling and accuracy. When we think about very high accuracy, physical limits also come into view: Molecules and atoms are very small, but not arbitrarily small. For what n does 10^{-n} metres become physically meaningless?

Hegel writes that there is an ascent from the general to the concrete (well explained in Siebert (2006, p. 159)). Therefore, it is not surprising for us that it can also be more difficult in mathematics lessons to establish and work out an increasingly concrete reference to reality than "only" practising abstract, i.e. content-empty algorithms. How much reference to reality does make sense? What can be learned from trying to get particularly close to reality when dealing with a simple question (2 + 2 = ?)? We see a useful learning opportunity here. In terms of content, the conceptual pair abstract/concrete is very important when thinking about reality. Mathematics plays an important role here (together with logic) as an example of successful abstraction with always astonishing concrete relevance in practice. Methodologically, there is a wonderful opportunity for self-determined learning. Learners can decide for themselves how far they want to go on the path of concretisation. Opportunities where learners in mathematics classes can decide for themselves on the course of instruction and the choice of topics (without risks to their success in the leaving examination) are extremely rare, but very valuable with regard to the central teaching goals of "independence" or "maturity" (Adorno, 1971).

6.5 Projects

These two central teaching goals ("independence", "maturity") will be better achieved if, with the help of the teacher, students also practise and learn to make responsible decisions in mathematics lessons. Reality-based projects offer particularly good opportunities for this, because after each turn in the spiral from WE to MODEL to REALITY and back to US, a decision has to be made: Is this enough? Do we continue? (Cf. Sections 4 and 5.)

In our contribution (to suggest just one example of many here) in ISTRON Volume 6 on the flight of the hawk, we deliberately elaborated such decision points (Maaß and Götz, 2019). From stage to stage, we need better data, new mathematical tools, etc., and are rewarded for the effort by becoming better and better at mathematically modelling and understanding the data from reality.

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